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AN INVESTIGATION OF PLASTIC TORSION IN CIRCULAR PRISMATIC BARS
WITH A CONCENTRIC OR ECCENTRIC HOLE USING THE
SAND-HEAP ANALOGY

A Thesis

by

GERALD JOHN CAREY, JR.

MAY 27 1963

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Submitted to the Graduate School of the
Agricultural and Mechanical College of Texas in
partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 1963

Major Subject: Aerospace Engineering

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GERALD JOHN CAREY, JR.

Approved as to style and content by:

(Chairman of Committee)

(Head of Department)

May 1963

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LIST OF SYMBOLS

x, y, z	Rectangular coordinates
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing-stress components
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	Shearing-strain components
$\epsilon_x, \epsilon_y, \epsilon_z$	Unit elongation in x-, y-, and z- directions
$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress parallel to x-, y-, and z- directions
u, v, w	Components of displacements
θ	Angle of twist per unit length
ϕ	Warping function
b	Number of basic dimensions
c	Offset distance from center of bar to center of hole
F	Elastic stress function
F_1	Plastic stress function
G	Modulus of elasticity in shear
h	Height of cylinder in sand-heap
I_2	Second invariant of the stress tensor
k	Yield shearing stress
m	Maximum slope of sand-heap
n	Number of dimensional quantities
p	Pressure acting on membrane
R	Radius of circular bar
r	Radius of circular hole
S	Area of hole

s	Surface tension developed in membrane
s_1	Number of non-dimensional π terms
V	Total volume
V_o	Volume of sand on tray
T	Torque
z	Vertical deflection of membrane

CHAPTER I

INTRODUCTION

In determining the shear stresses developed in a circular shaft by the action of a twisting moment, the strength of materials approach provides an exact solution. One major assumption in this analysis is that plane sections normal to the axis before twisting remain so after twisting.

This assumption is not valid for a non-circular prismatic bar or any cross section with a non-concentric circular hole. The solution to this problem is more difficult and considered in the theory of elasticity. As will be shown, the theory of elasticity embodies a solution assuming a linearly elastic material and involving the introduction of a stress function, the appropriate derivatives of which define the shearing stresses. If the particular stress function satisfies the conditions of equilibrium, boundary conditions, and compatibility, the solution is mathematically unique.

As the torque is increased, material will pass from the elastic to the plastic state. There is a limit to the ability of the cross section to withstand increasing torque. This limit is called the plastic strength in torsion; it represents that value of torque above which there is no resistance to flow and complete failure results with continued deformation. At this condition torque is no longer a function of rotation.

The purpose of this research was to investigate and predict the plastic strength in torsion of circular prismatic bars with a concentric or eccentric hole.

CHAPTER II

THEORY OF TORSION ANALOGIES

The Membrane Analogy for Elastic Torsion

Consider a cylindrical or prismatic bar of constant cross section which is twisted and held in equilibrium by twisting moments applied at its ends. The cross section of such a bar is shown in Figure 1. The bar is considered to be composed of an isotropic material possessing the idealized stress-strain relationship for an elastic, perfectly plastic material shown in Figure 2.

Increasing torque causes the material to pass from the elastic region (line AB, Figure 2) into the perfectly plastic range (line BC, Figure 2). After a point in the cross section reaches the yield stress in shear (point B), this maximum shearing stress remains a constant value k as increasing torque causes an increase in the plastic region of the bar. Before examining the plastic behavior of the prismatic cross section, consider the stress characteristics in the elastic range.

The exact solution to the problem of elastic torsion of prismatic bars was first postulated by Saint-Venant in 1855.^{1*} The displacements in the x and y directions due to rotation of the cross sections are designated by u and v , respectively. The warping of the cross section is denoted by w and defined by the function ϕ . These displacements are assumed to be:²

* References cited are listed numerically at the end of this thesis.

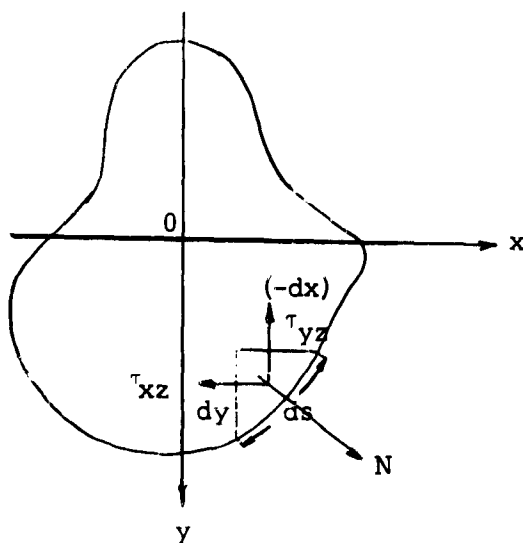


Figure 1. Two-Dimensional View, Prismatic Cross Section Subjected to Torsion.

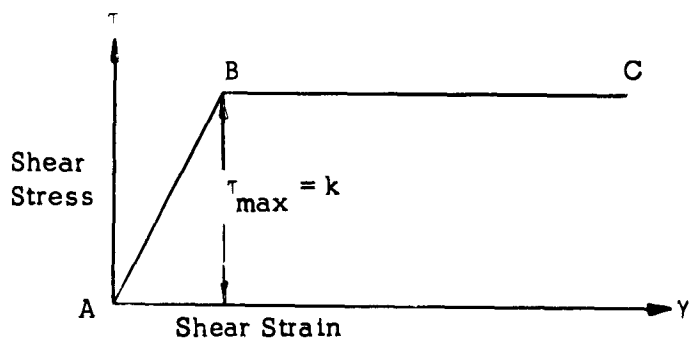


Figure 2. Idealized Shearing Stress-Strain Curve

$$\begin{aligned}
 u &= -\theta zy \\
 v &= \theta zx \\
 w &= \theta \phi(x, y)
 \end{aligned} \tag{1}$$

With reference to Figure 3, θz is the angle of rotation of the cross section at a distance z from the origin 0.

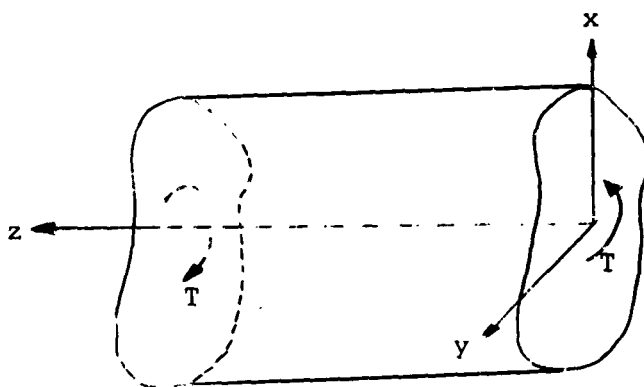


Figure 3. Three-Dimensional View, Prismatic Cross Section Subjected to Torsion

In terms of the assumed displacements u , v , and w , the components of strain are³

$$\begin{aligned}
 \epsilon_x &= \epsilon_y = \epsilon_z = \gamma_{xy} = 0 \\
 \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta \left(\frac{\partial \phi}{\partial x} - y \right) \\
 \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta \left(\frac{\partial \phi}{\partial y} + x \right)
 \end{aligned} \tag{2}$$

The corresponding stresses as given by elasticity theory are

$$\begin{aligned}\sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \\ \tau_{xz} &= G \gamma_{xz} = G \theta \left(\frac{\partial \theta}{\partial x} - y \right) \\ \tau_{yz} &= G \gamma_{yz} = G \theta \left(\frac{\partial \theta}{\partial y} + x \right)\end{aligned}\quad (3)$$

As pointed out by Timoshenko,⁴ from the assumed displacements, there will be no normal stresses acting between the longitudinal fibers of the shaft or in the longitudinal directions of these fibers. At each point in the cross section a condition of pure shear, as defined by the stress tensor of (4) below, exists.

$$\begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix}\quad (4)$$

From consideration of the equilibrium of forces acting on a rectangular parallelepiped of sides dx , dy , and dz , the equilibrium equations, neglecting body forces, reduce to⁵

$$\begin{aligned}\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} &= 0 \\ \frac{\partial \tau_{xz}}{\partial z} &= 0, \quad \frac{\partial \tau_{yz}}{\partial z} = 0\end{aligned}\quad (5)$$

Equation (3) satisfies the last two conditions of Equation (5).

The vital step in the analysis, according to Den Hartog,⁶ is the assumption by Saint-Venant that there is a function $F(x,y)$ such that the stresses may be found from it by differentiation, that is:

$$\tau_{xz} = \frac{\partial F}{\partial y}, \quad \tau_{yz} = -\frac{\partial F}{\partial x} \quad (6)$$

The expression $F(x,y)$ is called the elastic stress function. Substitution of the expressions of Equations (6) into (3) yields

$$\frac{\partial F}{\partial y} = G\theta \left(\frac{\partial \theta}{\partial x} - y \right), \quad -\frac{\partial F}{\partial x} = G\theta \left(\frac{\partial \theta}{\partial y} + x \right) \quad (7)$$

Differentiating the first expression of Equations (7) with respect to y , the second expression with respect to x , and subtracting from the first yields the following partial differential equation in terms of the elastic stress function F

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -2G\theta \quad (8)$$

Equation (8) must satisfy the boundary condition that the resultant shearing stress at the boundary is directed along the tangent to the boundary,⁷ that is

$$\tau_{xz} \frac{dy}{ds} + \tau_{yz} \left(-\frac{dx}{ds} \right) = 0 \quad (9)$$

where (see Figure 1), $\cos(N,x) = \frac{dy}{ds}$ and $\cos(N,y) = -\frac{dx}{ds}$. In terms of the elastic stress function F , Equation (9) becomes

$$\frac{\partial F}{\partial y} \frac{dy}{ds} + \frac{\partial F}{\partial x} \frac{dx}{ds} = \frac{dF}{ds} = 0 \quad (10)$$

Since the total derivative of the elastic stress function F with respect to the boundary distance s is zero, the elastic stress function F is constant along the boundary of the cross section.

The twisting moment T , with reference to Figure 1, is expressed by the double integral

$$T = \iint (\tau_{yz} x - \tau_{xz} y) dx dy \quad (11)$$

If the constant value of the stress function F on the boundary curve $y=f(x)$ of the cross section is taken equal to zero, as may be done for singly connected boundaries, integration of Equation (11) yields⁸

$$T = 2 \iint F dx dy \quad (12)$$

As Murphy indicates,⁹ the actual determination of the stress function as defined by Equation (8) for all but a few relatively simple cross sections becomes difficult, if not impossible, to determine analytically.

Quantitative solutions to the differential Equation (8), however, can be obtained by use of the membrane analogy. Prandtl observed in 1903 that the equation (8) of the elastic stress function is the same as the differential equation for the shape of a stretched membrane, originally flat, which is inflated by air pressure from below. This equation, as derived from the equilibrium of forces when summed in the direction z , is¹⁰

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = - \frac{p}{s} \quad (13)$$

In Equation (13) z is the deflection of the membrane, p is the pressure acting on the membrane, and s is the surface tension developed in the membrane. Equation (8), the differential equation of elastic torsion, is identical in form to Equation (13) if

$$\frac{p}{s} = 2G\theta \quad (14)$$

If in addition the boundary conditions are satisfied, the deflection z will equal $F(x,y)$, the elastic stress function. In summary, the stress function $F(x,y)$ is represented by the curved surface of the membrane and the contour lines of the surface represent the stress lines of the elastically twisted cross section. The tangent in the xy plane to any contour line of the membrane is in the direction of the resultant shearing stress τ . The stress is proportional to the slope of the stress surface $F(x,y)$. The twisting moment T is equal to twice the volume enclosed by the stress surface.

These characteristics of the membrane analogy have permitted extensive quantitative measurements of the stresses in a twisted bar within the elastic range. Griffith¹¹ and Hetenyi¹² outline applications of the membrane analogy.

The Sand-Heap Analogy for Plastic Torsion

If the cross section of Figure 1 is twisted beyond the yield point, certain parts of the bar will be deformed plastically. The condition at which

plastic deformation commences is, according to Prager,¹³ expressed by

$$I_2 - k^2 = 0 \quad (15)$$

where I_2 is the second invariant of the stress tensor. It is independent of the orientation of the coordinate axes. I_2 is expressed by (see Reference 14)

$$I_2 = -(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \quad (16)$$

The condition of stress in torsion as represented by the stress tensor of (4) produces, when substituted in the condition of plasticity of Equation (15),

$$\tau_{xz}^2 + \tau_{yz}^2 = k^2 = \text{constant} \quad (17)$$

The yield condition of Equation (15) was first postulated by Von Mises¹⁵ but its most widely known physical interpretation is embodied in the octahedral shearing stress theory introduced by Nadai.¹⁶

The equilibrium equations (5) must be satisfied in the plastic as well as elastic region. As noted by Nadai¹⁷, Equations (5) are satisfied if

$$\tau_{xz} = \frac{\partial F_1(x,y)}{\partial y}, \quad \tau_{yz} = -\frac{\partial F_1(x,y)}{\partial x} \quad (18)$$

in which $F_1(x,y)$ represents the plastic stress function of the cross section.

In those regions of the cross section where plastic flow occurs the condition of plasticity of Equation (17), in terms of the plastic stress function,

is

$$\left(\frac{\partial F_1}{\partial x}\right)^2 + \left(\frac{\partial F_1}{\partial y}\right)^2 = k^2 \quad (19)$$

The left side of Equation (19) is the square of the largest slope of the surface F_1 . From the mathematical definition of the gradient, Equation (19) may be expressed as

$$|\text{grad } F_1| = k = \text{constant} \quad (20)$$

where k is the maximum shearing stress shown in Figure 2.

As in the case of the elastic stress function F , the shearing stresses are directed tangentially to the contour lines of the plastic stress surface. The surface representing this plastic stress function may be thought of as the maximum value that the elastic stress function can attain. The plastic stress function may be considered as a "roof" under which the membrane, geometrically the same as the cross section, expands. When the membrane touches the "roof," the condition of plasticity of Equation (17) is satisfied and plastic yielding begins at that point in the cross section. As the torque is increased, as represented by increasing air pressure on the membrane, the membrane expands and touches more of the "roof." At the limit the membrane fills the entire volume under the roof. The cross section is considered to have attained a fully plastic state. Following the membrane analogy, the torque required to achieve the fully plastic state is equal to twice the volume under the "roof."

The mathematical and physical interpretation of the plastic stress function $F_1(x,y)$ for the case of complete yielding of the entire prismatic bar can be demonstrated by sand heaps covering a plate similar in cross section to the twisted bar. This analogy was first presented by Nadai at

a meeting of the German Society of Applied Mathematics and Mechanics in Marburg, Germany in 1923.

As described by Sadowsky,¹⁸ "...A plate whose shape is geometrically similar to the cross section of the twisted bar serves as a horizontal tray to hold a heap of dry uniform sand. The heap is to be as big as it is possible to pile by pouring a gentle stream of sand on top of the model, the excessive sand rolling freely down the slopes of the heap and falling off the elevated tray. The maximum-size heap being formed, its volume V is determined by weighing..." The twisting moment T in plastic torsion is now given by the equation

$$T = 2 \frac{k}{m} V \quad (21)$$

in which k is the maximum value of the shearing stress in Figure 2 and m is the maximum value of the slope of the lateral surface of the sand heap. In addition, the horizontal lines of the heap, projected vertically down on the supporting tray, give the system of trajectories of the maximum shearing stress k acting upon the cross section.

As pointed out by Murphy,¹⁹ the relative strength of cross sections of different shapes may be determined through use of the sand-heap analogy. It is the purpose of this investigation to evaluate the effect of circular holes, of varying size and location, upon the capability of a circular cross section to sustain plastic torsion. The sand-heap analogy as developed by Nadai and as modified by Sadowsky to include holes will be used.

The Sand-Heap Analogy Applied to Cross Sections with Holes

The twisting moment T , as given by Equation (21) is applicable to cross sections with and without holes. However, in carrying out the experiment it can be visualized that even the presence of a small hole in the cross section will permit an excessively large amount of sand to flow through it. The volume of sand remaining on the tray would not therefore be proportional to the torque required to obtain a fully plastic condition in the cross section.²⁰

Therefore, Sadowsky in 1941 modified Nadai's original analogy. If the cross section has holes, similar holes must be made in the supporting tray. In order to prevent the sand from flowing out through these holes, cylinders capable of vertical movement must be inserted into the holes. To hold the tray and to insure vertical travel a device as shown in Figure 4 and Plate 1 was constructed. Trays A and B are geometrically similar to the cross section under investigation.

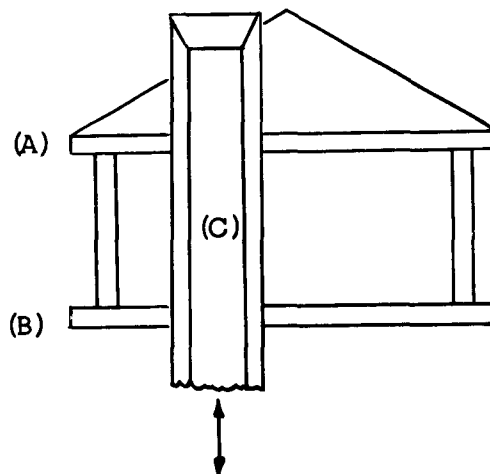


Figure 4. Model for Sand-Heap Analysis of Cross Section with Hole

It is important that the top of cylinder C be kept horizontal and its inner conical edges be sharpened so that a grain of sand placed on this edge will fall through the cylinder.

At the beginning of the experiment, the cylinder C is moved into a position as high as possible and a maximum sand heap is formed on the upper tray only. At this stage the cylinder is free of sand standing above the sand heap. After the maximum heap is formed, as shown in Figure 4, the cylinder is gradually lowered. At a certain position the top of the cylinder will reach the surface of the heap and the sand will commence to fall through the top of the cylinder. The lowering of the cylinder will continue until the brim of the cylinder is completely submerged in the sand. In the final position, the cylinder is at the maximum height at which no part of its lateral surface will protrude from the heap and be visibly exposed. Although extremely complicated mathematical surfaces may be formed in three-dimensional sand heaps, the geometric problem solves itself through the medium of the sand. This procedure applies as well to cross sections with more than one hole.

Designate V_0 as the volume of the sand remaining on the tray after the cylinder or cylinders have been lowered. Let $S_1, S_2 \dots S_n$ be the areas of the holes, with depths $h_1, h_2, \dots h_n$ - as measured from the level of the tray up to the top of the cylindrical opening. Then the volume V as defined by Equation (21) is

$$V = V_0 + S_1 h_1 + S_2 h_2 + \dots S_n h_n \quad (22)$$

The twisting moment T is therefore given by

$$T = 2 \frac{k}{m} (V_0 + S_1 h_1 + S_2 h_2 + \dots S_n h_n) \quad (23)$$

The mathematical derivation of Equation (23) follows. Consider the twisted prismatic cross section with a hole shown in Figure 5.

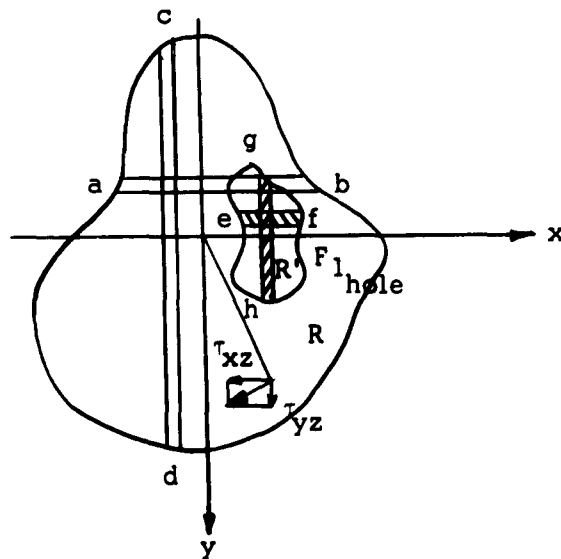


Figure 5. Two-Dimensional View, Prismatic Cross Section with Hole.

R is the entire cross-sectional area including the region R' of the hole.

The twisting moment T is therefore

$$T = \iint_{R(x,y)} (\tau_{yz}x - \tau_{xz}y) dx dy - \iint_{R'(x,y)} (\tau_{yz}x - \tau_{xz}y) dx dy \quad (24)$$

Defining the shearing stresses in terms of the plastic stress function F_1 , one obtains

$$\tau_{xz} = \frac{k}{m} \frac{\partial F_1}{\partial y} \text{ and } \tau_{yz} = -\frac{k}{m} \frac{\partial F_1}{\partial x} \quad (25)$$

Substituting these expressions for shearing stresses into Equation (24) and noting that k and m are constants, one finds

$$T = -\frac{k}{m} \iint_R \left(\frac{\partial F_1}{\partial x} x + \frac{\partial F_1}{\partial y} y \right) dx dy + \frac{k}{m} \iint_{R'} \left(\frac{\partial F_1}{\partial x} x + \frac{\partial F_1}{\partial y} y \right) dx dy$$

Consider the first integral over the region R with the limits of integration indicated from Figure 5,

$$\begin{aligned} -\frac{k}{m} \iint_R \left(\frac{\partial F_1}{\partial x} x + \frac{\partial F_1}{\partial y} y \right) dx dy &= -\frac{k}{m} \left\{ \int_c^d \left[\int_a^b \frac{\partial F_1}{\partial x} x dx \right] dy \right. \\ &\quad \left. + \int_a^b \left[\int_c^d \frac{\partial F_1}{\partial y} y dy \right] dx \right\} = -\frac{k}{m} \left\{ \int_c^d \left[x F_1 \Big|_a^b - \int_a^b F_1 dx \right] dy \right. \\ &\quad \left. + \int_a^b \left[y F_1 \Big|_c^d - \int_c^d F_1 dy \right] dx \right\} = -\frac{k}{m} \left\{ \int_c^d \left[x_b F_{1_b} - x_a F_{1_a} \right] dy \right. \\ &\quad \left. - \int_c^d \int_a^b F_1 dx dy + \int_a^b \left[y_d F_{1_d} - y_c F_{1_c} \right] dx - \int_a^b \int_c^d F_1 dy dx \right\} \end{aligned}$$

However, $F_{1_a} = F_{1_b} = F_{1_c} = F_{1_d} = F_1$ outside; therefore, the first integral equals

$$\frac{k}{m} \left\{ 2 \int_a^b \int_c^d F_1 dy dx - F_{1_a} \int_c^d (x_b - x_a) dy - F_{1_a} \int_a^b (y_d - y_c) dx \right\}$$

$$\text{But } \int_c^d (x_b - x_a) dy = \int_a^b (y_d - y_c) dx = A, \text{ the area of region R.}$$

Therefore, the first integral equals

$$\frac{k}{m} \left\{ 2 \int_a^b \int_c^d F_1 dy dx - 2 F_{1_{\text{outside}}} A \right\}$$

Taking $F_{1_{\text{outside}}}$ equal to zero, which is the height of the base of the sand heap, one obtains from the first integral

$$2 \frac{k}{m} \int_a^b \int_c^d F_1 dy dx$$

In a similar manner the second integral over R' is

$$\frac{k}{m} \iint_{R_1} \left(\frac{\partial F_1}{\partial x} x + \frac{\partial F_1}{\partial y} y \right) dx dy = -2 \frac{k}{m} \int_e^f \int_g^h F_1 dy dx + 2 \frac{k}{m} F_{1_{\text{hole}}} A'$$

where A' is the area of the region R' . Since the top of the region R' is made open, the height of the sand heap in R' is zero and F_1 is zero over the region R' . The second integral therefore equals $2 \frac{k}{m} F_{1_{\text{hole}}} A'$.

The twisting moment T is

$$T = 2 \frac{k}{m} \left\{ \int_a^b \int_c^d F_1 dy dx + F_{1_{\text{hole}}} A' \right\} \quad (26)$$

Since F_1 is equal to the height of the sand heap, the terms in braces are the volume of the sand on the tray V_0 and the volume enclosing the hole, respectively. If $F_{1_{\text{hole}}}$ is designated as h_1 , the height of the

sand heap at the hole, it follows that the twisting moment T as expressed by Equation (23) and Equation (26) are equal for a cross section with one hole.

For this case

$$T = 2 \frac{k}{m} \left\{ V_o + h_1 S_1 \right\} \quad (27)$$

This relationship will be used directly for this analysis.

CHAPTER III

DIMENSIONAL ANALYSIS

In order for the results to be of general application, use was made of the Buckingham π theorem to present the experimental results in non-dimensional form. The Buckingham π theorem states that the number of dimensionless and independent quantities required to express a relationship among the variables in any phenomenon is equal to the number of quantities involved minus the number of dimensions in which these quantities may be measured. That is, $s_1 = n - b$, where s_1 is the number of π terms, n is the total number of quantities involved, and b is the number of basic dimensions.²¹

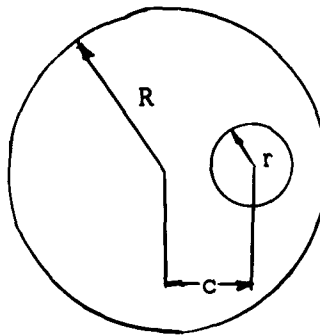


Figure 6. Two-Dimensional View, Circular Cross Section with Circular Hole.

For this case (see Figure 6) choose

$$T = f(k, R, r, c) \quad (1a)$$

Dimensionally

$$T = \text{twisting moment} \doteq (FL)$$

$$k = \text{maximum shearing stress} \doteq (FL^{-2})$$

$$R = \text{radius of circular cross section} \doteq (L)$$

$$r = \text{radius of circular hole} \doteq (L)$$

$$c = \text{distance from center of hole to center of cross section} \doteq (L)$$

Since

$$s_1 = n - b = 5 - 2 = 3$$

Three π terms are necessary to describe twisting moment behavior of the cross section in plastic torsion.

From Equation (1a) where \doteq denotes dimensional equivalence

$$1 \doteq (T)^a (r)^b (c)^c (R)^d (k)^e$$

Dimensionally

$$1 \doteq (FL)^a (L)^b (L)^c (L)^d (FL^{-2})^e$$

$$F: 0 \doteq a + e$$

$$L: 0 \doteq a + b + c + d - 2e$$

Since there are two independent equations in five unknowns, evaluate d and e in terms of a , b , and c :

$$e = -a$$

$$d = -3a - b - c$$

Therefore

$$1 \doteq (T)^a (r)^b (c)^c (R)^{-3a-b-c} (k)^{-a}$$

$$1 \doteq \left(\frac{T}{R^3 k} \right)^a \left(\frac{r}{R} \right)^b \left(\frac{c}{R} \right)^c$$

Expressed functionally,

$$\left(\frac{T}{R^3 k} \right) = f \left[\left(\frac{r}{R} \right), \left(\frac{c}{R} \right) \right]$$

The experimental results will therefore be expressed in the three non-dimensional π terms of $\frac{T}{R^3 k}$, $\frac{r}{R}$, $\frac{c}{R}$.

CHAPTER IV

EXPERIMENTAL PROCEDURES

Prior to investigating the effect of a circular hole on the plastic strength in torsion of a circular section, sections whose plastic stress function equation were known were investigated. Circular, square, and rectangular cross sections were tested to verify experimental techniques. The results are contained in the Appendix.

The sand used was Ottawa sand, American Society Testing Materials designation C-190, produced by the Ottawa Silica Company, Ottawa, Illinois. Of uniform composition its specific volume was determined as the arithmetic mean of ten measurements to be 39.398 cubic inches per kilogram.

Its slope m was determined to be 0.636. The volume V_0 was determined by measuring the mass of sand remaining on the tray. The height of the circular cylinder h was also measured.

The value of $\frac{T}{kR^3}$ was determined using Equation (27) in terms of the sand-heap analogy. In order to measure the effect of varying size and location of the hole, the radius of the hole was varied as follows: $r = 0.25$ in., 0.50 in., 0.75 in., 0.875 in., 1.125 in., 1.313 in., 1.50 in., 1.750 in., 2.0 in., 2.250 in., and 3.313 in. The radius of the circular cross section R was constant at six in. The offset c varied as follows: $c = 1.2$ in., 2.4 in., 3.6 in., 4.8 in., and 6.0 in.

Plates 1 through 8 (see Appendix) picture the model and several sand heaps formed.

CHAPTER V

RESULTS AND CONCLUSIONS

Results are shown in Figures 7 and 8. The data for each size hole and keyway are tabulated in Tables 1 through 11. Figure 7 shows the maximum strength in torsion when the circular shaft has an interior hole. Figure 8 predicts the maximum strength in torsion of a circular shaft with a keyway. It is noted in the tables that when the offset distance c has increased to where the hole has become a keyway, the region becomes singly connected and the stress function is taken as zero about the keyway circumference.

The curves are used in the following manner. Knowing the size r , location c , and radius of the cross section R , the value $\frac{T}{kR^3}$ is determined if $\frac{r}{R} \leq 0.56$. The yield stress k is determined from the shearing stress-strain curve of the material. The value of T , the maximum strength in torsion, is then determined in the appropriate dimensions of k and R .

With the exception of mild steel, which behaves very nearly as a perfectly plastic material at the beginning of strain hardening,²² the material possessing the characteristics of Figure 2 are rarely found. However, depending on the degree of conservativeness desired in the design and the specific material characteristics, an idealization of the shearing stress-strain curve can be made to determine the value k .

The above analysis assumes that the shearing strains remain small in order that no components of normal stresses act in sections normal or parallel to the axis of the bar.²³

Mindful of these assumptions and resulting limitations, Figures 7 and 8 can be used in predicting the maximum strength in torsion of circular cross sections with holes.

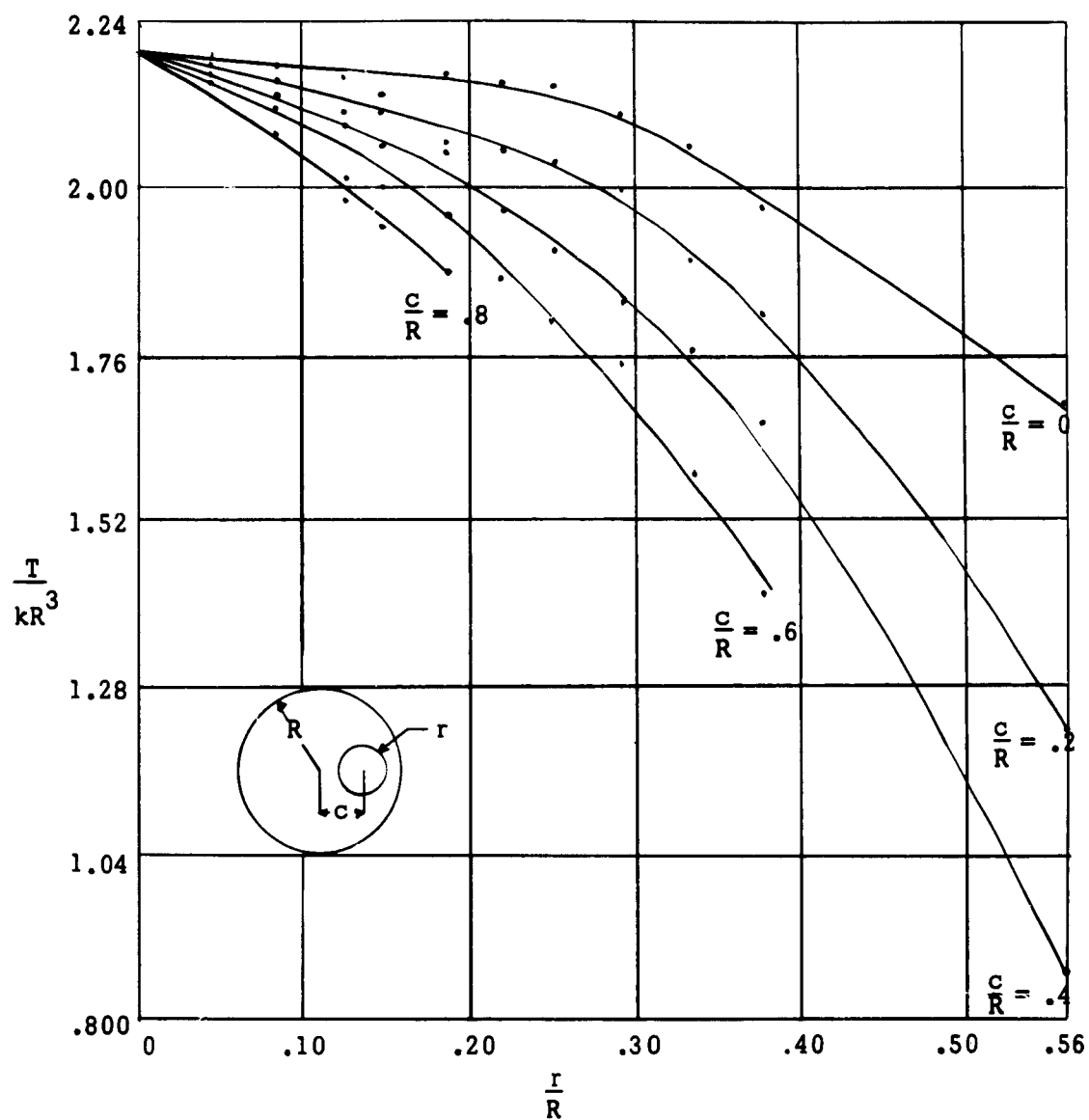


Figure 7. Maximum Strength in Torsion—Circular Bar with Hole.

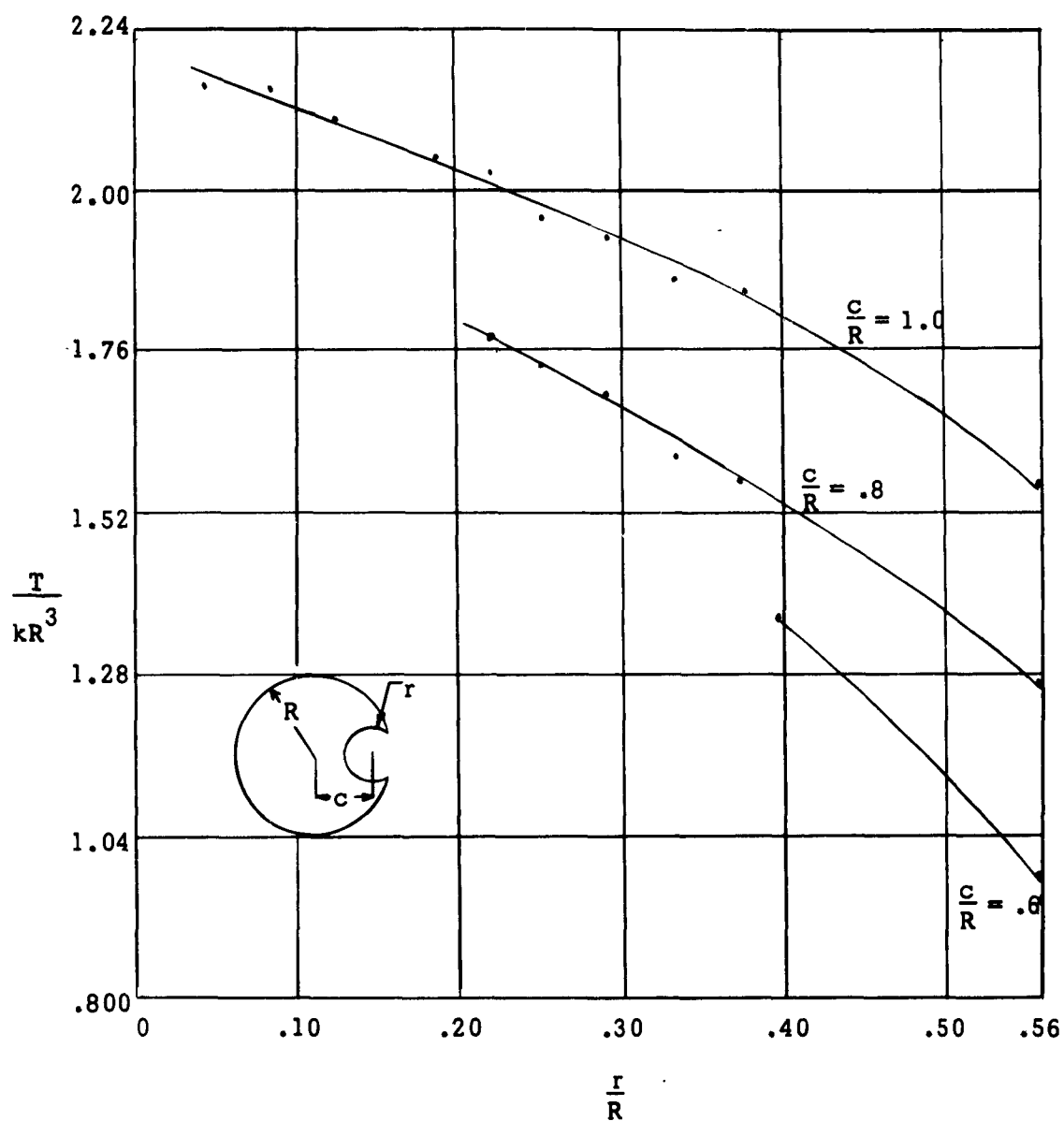


Figure 8. Maximum Strength in Torsion-Circular Bar with Keyway.

Table 1. Experimental Results, * $\frac{r}{R} = 0$, $\frac{r}{R} = 0.0417$, $r = 0.25$ "

$\frac{c}{R}$	Mass (gms)	V_0 (in ³)	h (in)	hg (in ³)	$V_0 + hg$	$\frac{T}{kR^3}$
no hole	3850	151.682			151.682	
	3821	150.540			150.540	2.1976
	3824	150.658			150.658	
0	3712	146.245	3.250	0.638	146.883	
	3726	146.797	3.156	0.619	147.416	2.1464
	3740	147.349	3.156	0.619	147.968	
0.2	3776	148.767	2.686	0.528	149.295	
	3778	148.846	2.500	0.491	149.337	2.184
	3828	150.816	2.906	0.570	151.386	
0.4	3827	150.776	2.031	0.399	151.175	
	3774	148.688	1.906	0.374	149.062	2.1816
	3777	148.806	2.013	0.399	149.205	
0.6	3745	147.546	1.188	0.233	147.779	
	3762	148.136	1.250	0.246	148.382	2.1672
	3817	150.382	1.219	0.239	150.406	
0.8	3744	147.506	0.500	0.098	147.614	
	3772	148.609	0.594	0.116	148.725	2.168
	3812	150.185	0.625	0.122	150.307	
1.0	3747	147.624				
	3768	148.452				2.1504
	3732	147.033				

* All calculations on Marchant Model SKA Desk Calculator.

Table 2. Experimental Results, $\frac{r}{R} = 0.0833$, $r = 0.50$ in.

$\frac{c}{R}$	Mass (gms)	V_o (in ³)	h (in)	h_s (in ³)	$V_o + h_s$	$\frac{T}{kR^3}$
0	3732	147.033	3.219	2.528	149.561	2.167
	3739	147.309	3.187	2.503	149.812	
	3672	144.670	3.157	2.480	147.150	
0.2	3732	147.033	2.469	1.939	148.972	2.174
	3732	147.033	2.594	2.037	149.070	
	3753	147.861	2.531	1.988	149.849	
0.4	3682	145.063	1.844	1.448	146.511	2.150
	3665	144.394	1.781	1.399	145.793	
	3707	146.048	1.875	1.473	147.521	
0.6	3701	145.812	1.157	0.909	146.721	2.122
	3647	143.685	1.157	0.909	144.594	
	3679	144.945	1.188	0.933	145.878	
0.8	3541	139.508	0.427	0.335	139.843	2.079
	3621	142.660	0.406	0.319	142.979	
	3687	145.260	0.379	0.295	145.555	
1.0	3737	147.230				2.148
	3739	147.309				
	3757	148.018				

Table 3. Experimental Results, $\frac{r}{R} = 0.125$, $r = 0.75$ in.

$\frac{c}{R}$	Mass (gms)	V_o (in ³)	h (in)	h_s (in ³)	$V_o + h_s$	$\frac{T}{kR^3}$
0	3637	143.290	3.000	5.301	148.591	2.164
	3625	142.818	3.031	5.356	148.174	
	3652	143.881	3.000	5.301	149.182	
0.2	3597	141.715	2.281	4.031	145.746	2.107
	3572	140.730	2.156	3.801	144.531	
	3548	139.784	2.313	4.087	143.871	
0.4	3572	140.730	1.594	2.817	143.547	2.089
	3586	141.281	1.594	2.817	144.098	
	3550	139.863	1.625	2.872	142.735	
0.6	3515	138.484	0.875	1.546	140.030	2.010
	3452	136.002	0.938	1.658	137.660	
	3422	134.820	0.906	1.601	136.421	
0.8	3452	136.002	0.188	0.332	136.334	1.982
	3397	133.835	0.219	0.387	134.222	
	3492	137.578	0.219	0.387	137.965	
1.0	3664	144.352				2.096
	3614	142.384				
	3687	145.260				

Table 4. Experimental Results, $\frac{r}{R} = 0.146$, $r = 0.875$ in.

$\frac{c}{R}$	Mass (gms)	V_o (in ³)	h (in)	h_s (in ³)	$V_o + h_s$	$\frac{T}{kR^3}$
0	3520	138.681	2.969	7.141	145.822	
	3577	140.926	2.875	6.916	147.842	2.1334
	3529	139.034	2.875	6.916	145.950	
0.2	3572	140.730	2.156	5.186	145.916	
	3560	140.257	2.344	5.638	145.895	2.120
	3538	139.390	2.344	5.638	145.028	
0.4	3542	139.948	1.531	3.683	143.631	
	3450	135.923	1.594	3.834	139.757	2.060
	3480	137.105	1.656	3.983	141.088	
0.6	3409	134.308	0.937	2.254	136.562	
	3467	136.593	0.937	2.254	138.847	2.006
	3444	135.687	0.906	2.180	137.867	
0.8	3442	135.608	0.188	0.452	136.060	
	3352	132.062	0.188	0.452	132.514	1.942
	3340	131.590	0.188	0.452	131.635	
1.0	3635	143.212				
	3567	140.533				2.074
	3645	143.606				

Table 5. Experimental Results, $\frac{r}{R} = 0.188$, $r = 1.125$ in.

$\frac{c}{R}$	Mass (gms)	V_O (in ³)	h (in)	h_s (in ³)	$V_O + h_s$	$\frac{T}{kR^3}$
0	3496	137.735	2.938	11.681	149.416	
	3477	136.987	2.938	11.681	148.668	2.166
	3472	136.790	2.875	11.431	148.221	
0.2	3502	137.972	2.500	9.940	147.912	
	3527	138.956	2.500	9.940	148.896	2.145
	3432	135.214	2.500	9.940	145.154	
0.4	3419	134.702	1.563	6.214	140.916	
	3492	137.578	1.594	6.338	143.916	2.067
	3426	134.978	1.531	6.087	141.065	
0.6	3394	132.717	0.750	2.982	136.689	
	3352	132.062	0.718	2.855	134.917	1.965
	3310	130.407	0.718	2.855	133.262	
0.8	3292	129.698	0.0312	0.124	129.822	
	3294	129.777	0.0312	0.124	129.901	1.884
	3257	128.319	0.0312	0.124	128.443	
1.0	3522	138.760				
	3602	141.911				2.049
	3592	141.517				

Table 6. Experimental Results, $\frac{r}{R} = 0.219$, $r = 1.313$ in.

$\frac{c}{R}$	Mass (gms)	V_o (in ³)	h (in)	hs (in ³)	$V_o + hs$	$\frac{T}{kR^3}$
0	3384	133.323	2.656	14.385	147.708	
	3362	132.456	2.750	14.894	147.350	2.150
	3385	133.362	2.687	14.553	147.915	
0.2	3285	129.422	2.031	11.000	140.422	
	3331	131.235	2.031	11.000	142.235	2.056
	3297	129.895	2.031	11.000	140.895	
0.4	3266	128.674	1.406	7.615	136.289	
	3237	127.531	1.406	7.615	135.146	1.972
	3232	127.334	1.406	7.615	134.949	
0.6	3182	125.364	0.656	3.553	128.917	
	3126	123.158	0.656	3.553	126.711	1.869
	3197	125.955	0.687	3.553	129.508	
0.8	3097	122.016				
	3132	123.395				1.791
	3137	123.592				
1.0	3564	140.414				
	3506	138.129				2.026
	3519	138.646				

Table 7. Experimental Results, $\frac{r}{R} = 0.250$, $r = 1.500$ in.

$\frac{c}{R}$	Mass (gms)	V_o (in ³)	h (in)	h_s (in ³)	$V_o + h_s$	$\frac{T}{kR^3}$
0	3292	129.699	2.625	18.555	148.254	
	3275	129.028	2.719	19.220	148.250	2.152
	3262	128.516	2.594	18.336	146.852	
0.2	3205	126.271	2.063	14.583	140.854	
	3177	125.167	2.000	14.137	139.304	2.044
	3209	126.428	2.063	14.583	141.011	
0.4	3117	122.804	1.313	9.281	132.085	
	3096	121.976	1.344	9.500	131.476	1.912
	3087	121.622	1.219	8.617	130.239	
0.6	3107	122.409	0.656	4.637	127.046	
	3042	119.849	0.594	4.199	124.048	1.810
	2994	117.958	0.563	3.980	121.938	
0.8	3022	119.061				
	3032	119.455				1.745
	3069	120.912				
1.0	3414	134.505				
	3395	133.756				1.960
	3442	135.608				

Table 8. Experimental Results, $\frac{r}{R} = 0.292$, $r = 1.750$

$\frac{c}{R}$	Mass (gms)	V_O (in ³)	h (in)	hs (in ³)	$V_O + hs$	$\frac{T}{kR^3}$
0	3054	120.321	2.406	23.149	143.470	
	3098	122.055	2.500	24.053	146.108	2.100
	3028	119.297	2.469	23.755	143.052	
0.2	2987	117.682	1.844	17.741	135.423	
	3071	120.991	1.865	17.944	138.935	1.998
	3022	119.061	1.906	18.338	137.399	
0.4	2917	114.924	1.094	10.526	125.450	
	2954	116.382	1.094	10.526	126.908	1.838
	2943	115.948	1.094	10.526	126.474	
0.6	2909	114.609	0.500	4.811	119.420	
	2922	115.121	0.500	4.811	119.932	1.745
	2929	115.397	0.500	4.811	120.208	
0.8	2968	116.933				
	2966	118.036				1.700
	2928	115.357				
1.0	3358	132.298				
	3370	132.771				1.930
	3364	132.535				

Table 9. Experimental Results, $\frac{r}{R} = 0.333$, $r = 2.0$ in.

$\frac{c}{R}$	Mass (gms)	V_0 (in ³)	h (in)	h_s (in ³)	$V_0 + h_s$	$\frac{T}{kR^3}$
0	2837	111.772	2.281	28.664	140.436	
	2824	111.260	2.250	28.274	139.534	2.038
	2823	111.221	2.281	28.664	139.885	
0.2	2793	110.039	1.687	21.200	131.239	
	2742	108.029	1.687	21.200	129.229	1.893
	2741	107.990	1.719	21.602	129.592	
0.4	2789	109.881	1.000	12.566	122.447	
	2772	109.221	1.000	12.566	121.787	1.772
	2758	108.660	0.969	12.177	120.837	
0.6	2692	106.059	0.375	4.712	110.771	
	2659	104.759	0.344	4.323	109.082	1.596
	2667	105.074	0.313	3.933	109.007	
0.8	2789	109.881				
	2838	111.812				1.613
	2807	110.590				
1.0	3287	129.501				
	3254	128.201				1.869
	3231	127.295				

Table 10. Experimental Results, $\frac{r}{R} = 0.375$, $r = 2.25$ in.

$\frac{c}{R}$	Mass (gms)	V_o (in ³)	h (in)	hs (in ³)	$V_o + hs$	$\frac{T}{kR^3}$
0	2637	103.893	2.156	34.290	138.183	
	2597	102.317	2.188	34.799	137.116	1.973
	2612	102.908	2.158	34.322	137.222	
0.2	2594	102.198	1.469	23.364	125.562	
	2570	101.253	1.469	23.364	124.617	1.820
	2571	101.292	1.469	23.364	124.656	
0.4	2527	99.559	0.906	14.409	113.968	
	2562	100.938	0.906	14.409	115.347	1.670
	2545	100.268	0.906	14.409	114.677	
0.6	2392	94.240	0.094	1.495	95.898	1.395
	2396	94.398	0.094	1.495	95.893	
0.8	2747	108.226				
	2740	107.950				1.575
	2750	108.345				
1.0	3223	126.980				
	3232	127.334				1.849
	3212	126.546				

Table 11. Experimental Results, $\frac{r}{R} = 0.5522$, $r = 3.313$ in.

$\frac{c}{R}$	Mass (gms)	V_o (in ³)	h (in)	hs (in ³)	$V_o + hs$	$\frac{T}{kR^3}$
0	1626	64.061	1.500	51.723	115.784	
	1645	64.810	1.500	51.723	116.533	1.752
	1654	65.164	1.500	51.723	116.887	
0.2	1562	61.540	0.625	21.551	83.091	
	1582	62.328	0.625	21.551	83.879	1.222
	1605	63.234	0.625	21.551	84.785	
0.4	1452	57.206	0.125	4.310	61.516	
	1415	55.748	0.125	4.310	60.058	0.880
	1405	55.354	0.125	4.310	59.664	
0.6	1727	68.040				
	1704	67.134				0.972
	1736	68.395				
0.8	2192	86.360				
	2221	87.503				1.270
	2232	87.936				
1.0	2723	107.281				
	2730	107.557				1.566
	2737	107.832				

APPENDIX



Plate 1 Model, $r/R = .375$, $c/R = 0$



Plate 2 Sand Heap, $r/R = .375$, $c/R = 0$



Plate 3 Sand Heap, $r/R = 0$, $c/R = 0$



Plate 4 Sand Heap, $r/R = .219$, $c/R = 1.0$

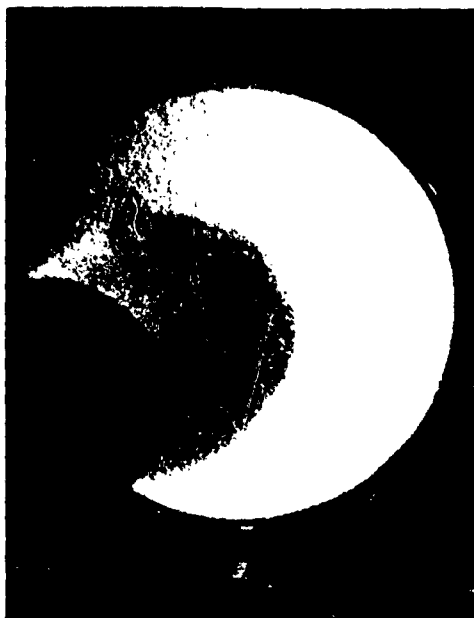


Plate 5 Sand Heap, $r/R = .552$, $c/R = 1.0$

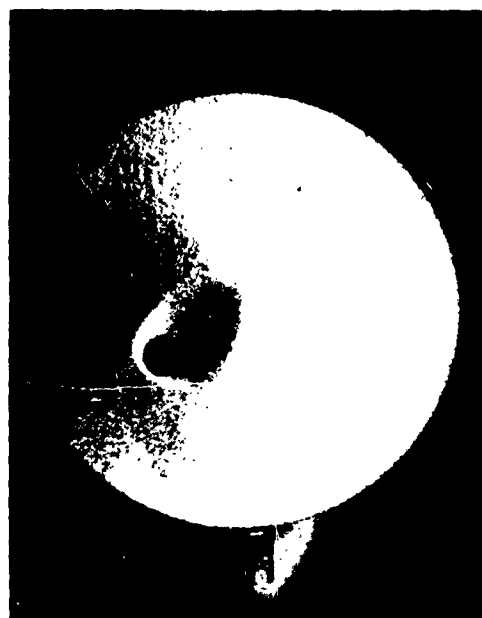


Plate 6 Sand Heap, $r/R = .0833$, $c/R = .4$

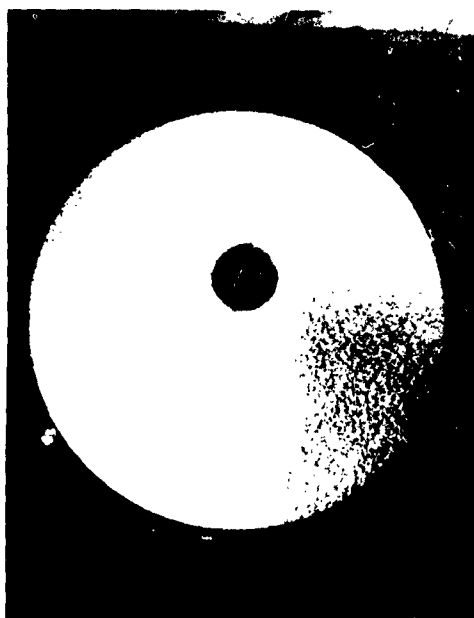


Plate 7 Sand Heap, $r/R = .292$, $c/R = .2$

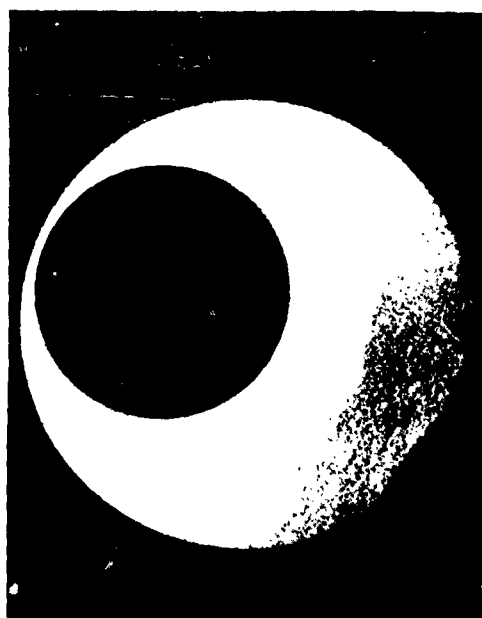
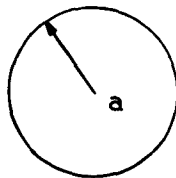


Plate 8 Sand Heap, $r/R = .552$, $c/R = .4$

CALIBRATION EXPERIMENTS

The plastic stress functions for such singly connected regions as a circular, square, and rectangular cross section can readily be predicted mathematically. The plastic stress function for a circular section is the surface of a right circular cone and its strength in plastic torsion is equal to twice the volume of a right circular cone whose slope is equal to the yield stress k . Similarly, the plastic strength of a square bar would equal twice the volume of a pyramid. In order to compare theoretical with experimental techniques, the above sections were tested and the value $\frac{T}{k}$ determined. The average per cent deviation from theory was 5.05 per cent.

Circular Section



$a = 5$ inches

Theory²⁴

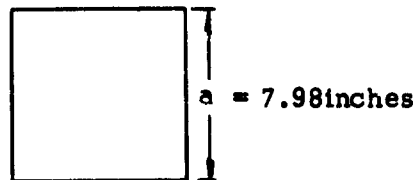
$$\frac{T}{k} = \frac{2\pi a^3}{3} = \frac{2\pi(125)}{3} = 261.5$$

Experiment

$$\frac{T}{k} = \frac{2V}{m}, \quad V = 82.8 \text{ cubic inches (determined by weighing)}$$

$$m = 0.636 \text{ (determined by measuring microscope)}$$

$$\frac{T}{k} = \frac{(2)(82.8)}{0.636} = 260.5. \text{ Therefore, per cent deviation from theory is } 0.384\%.$$

Square Section

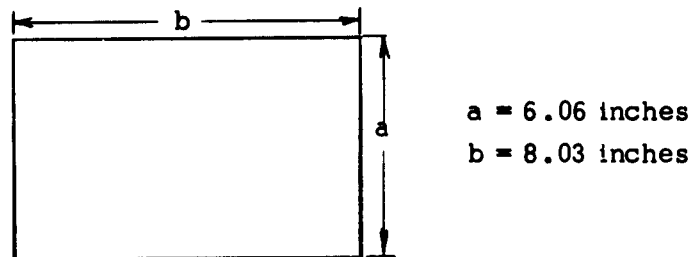
Theory $\frac{T}{k} = \frac{a^3}{3} = \frac{(7.98)^3}{3} = 169.7$

Experiment $\frac{T}{k} = \frac{2V}{m}$

$V = 57.5$ cubic inches

$$\frac{T}{k} = \frac{(2)(57.5)}{0.636} = 181$$

Per cent deviation from theory is 6.67 %.

Rectangular Section

Theory $\frac{T}{k} = \frac{a^3}{3} + \frac{a^2(b-a)}{2}$

$$\frac{T}{k} = \frac{(6.06)^3}{3} + \frac{(6.06)^2 [8.03 - 6.06]}{2} = 100.5$$

Experiment $\frac{T}{k} = \frac{2}{m} V = \frac{2}{0.636} (34.7) = 109$

Per cent deviation from theory is 8.1%.

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